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A model of strongly correlated electrons with condensed resonating-valence-bond ground states

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Abstract

We propose a new exactly solvable model of strongly correlated electrons. The model is based on a d–p model of the CuO_2 plane with infinitely large repulsive interactions on Cu-sites, and it contains additional correlated-hopping, pair-hopping and charge–charge interactions of electrons. For even numbers of electrons less than or equal to 2/3-filling, we construct the exact ground states of the model, all of which have the same energy and each of which is the unique ground state for a fixed electron number. It is shown that these ground states are the resonating-valence-bond states which are also regarded as condensed states in which all electrons are in a single two-electron state. We also show that the ground states exhibit off-diagonal long-range order.

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1. Introduction

Considerable attention has been paid to strongly correlated-electron systems in the study of high-T_c superconductivity since Anderson [1] proposed that its origin may be attributed to magnetic interactions induced by strong Coulomb repulsion. In Anderson's scenario, a Mott insulating phase of an undoped cuprate is assumed to be the resonating-valence-bond (RVB) state, composed of electron spin-singlet pairs, and these singlet pairs become charged superconducting pairs when the Mott insulator is doped sufficiently. Although many attempts have been made on lattice models of electrons with interactions, such as the Hubbard model and the t–J model, there is no definite answer to the question of whether this scenario is realized in these concrete microscopic models. These models are usually difficult to analyse theoretically, and exact results are limited except for the one-dimensional case where the Bethe ansatz is available.

One of the exact results in two or more dimensions is obtained by Brandt and Giesekus [2], who constructed the exact ground state of the Hubbard model with infinitely large repulsive

on-site interactions on a d-dimensional decorated hypercubic lattice. Tasaki [3] generalized the Brandt–Giesekus model to a class of Hubbard models on certain cell structures. He pointed out that the exact ground states in this class of models are regarded as RVB states. He also obtained the singlet pair correlation function on the tree lattice and discussed a possibility of long-range order associated with the singlet pairs and superconductivity. Bares and Lee [4] and Yamanaka *et al* [5] treated one-dimensional versions of the models and obtained various correlation functions, including the singlet pair correlation function. They found that all the correlation functions decay exponentially with distance and concluded that the system is in an insulating phase. There are no conclusive results about the occurrence (or absence) of superconductivity in the models in two or more dimensions.

For extended Hubbard models including nearest-neighbour interactions in addition to the on-site interactions, there are some exact results relevant to superconductivity. These results are based on the so-called η-pairing mechanism originally proposed by Yang [6]. By using the η-pairing mechanism Yang constructed eigenstates of the Hubbard model on the hypercubic lattice and proved that these states, in which all electrons form the η-pairs, have off-diagonal long-range order and thus are superconducting. Unfortunately, these eigenstates were shown not to be the ground states. But, Essler *et al* [7] showed that the η-pairing states become the exact ground states of an extended Hubbard model with attractive on-site interactions. They also showed that the model with moderately repulsive on-site interactions has ground states in which part of electrons forms the η-pairs. Possibilities of superconductivity due to the η-pairing mechanism were discussed in other related models, in particular, in one-dimensional systems. See, for example, [8–10] and references therein for more information.

Another rigorous result is a theorem on the Hubbard model with attractive on-site interactions on bipartite lattices due to Shen and Qiu [11]. They proved that these models exhibit off-diagonal long-range order in the ground states. See also [12] for related results.

At the present time, to draw an exact result about superconductivity in the Hubbard model with on-site repulsion or the t–J model seems to be a formidably difficult task, but we have a chance to obtain one in extended Hubbard models. Although these exactly solvable models have somewhat artificial aspects, they will give us some insight into mechanisms for the phenomenon. In this paper, motivated by this viewpoint, we propose a new exactly solvable model of strongly correlated electrons. It is based on a two-dimensional d–p model, a tight binding model of electrons with on-site interactions on the decorated square lattice corresponding to the CuO_2 plane. The present new model has infinitely large repulsive interactions on Cu-sites and furthermore contains correlated-hopping, pair-hopping and attractive charge–charge interactions of electrons. For even numbers of electrons we obtain the exact ground states of the model. A mechanism used to construct the ground states is similar to the η-pairing mechanism, but an electron pair used here is different from η-pair. Our electron pair consists of singlet pairs of electrons on the same O-site, on the nearest-neighbour pair of the Cu-sites, and on the nearest-neighbour pair of the Cu- and O-sites. The constructed exact ground states are expressed as a linear combination of products of these singlet pairs and thus are regarded as RVB states. We also show that these ground states exhibit off-diagonal long-range order, which allows us to construct a ground state with explicit electron-number symmetry breaking.

The recent experimental results of angle-resolved photoemission spectroscopy [13] suggested that electron–phonon coupling is important in high-T_c superconductors and thus a microscopic theory should include this effect. Although this fact does not immediately justify the addition of artificial interactions to a model Hamiltonian, we think that it is important to know what kind of electron–electron interaction, of Coulomb repulsion or electron–phonon

coupling (or others), induces this, may stabilize a superconducting state in a theoretical point of view. We hope that our results will be useful in a future study.

In the next section we define the model and state the main result as a theorem. In section 3, we give some remarks, including the RVB representation and off-diagonal long-range order. In section 4 we prove the theorem.

2. Definition of the model and the main result

We start by describing lattice Λ on which our model will be defined. Let L be an arbitrary positive integer and D be a set of sites

$$D = \{x = (x^{(1)}, x^{(2)}) \mid x^{(l)} \in \mathbb{Z}, 1 \leqslant x^{(l)} \leqslant L \text{ for } l = 1, 2\} \tag{2.1}$$

with periodic boundary conditions. Let P be a set of sites located at the mid-points of nearest-neighbour sites in D,

$$P = \{u = x + \delta_l/2 \mid x \in D, l = 1, 2\} \tag{2.2}$$

where δ_l is the unit vector along the l-axis. Then we define Λ as $\Lambda = D \cup P$, which mimics the CuO_2 plane. For later use, we also define

$$P_x = \{u \mid u \in P, |u - x| = 1/2\} \tag{2.3}$$

for $x \in D$ and

$$D_u = \{x \mid x \in D, |x - u| = 1/2\} \tag{2.4}$$

for $u \in P$.

Let $c_{i,\sigma}$ and $c^\dagger_{i,\sigma}$ be the annihilation and the creation operators, respectively, for an electron with spin σ at site $i \in \Lambda$. These operators satisfy the usual anticommutation relations

$$\{c_{i,\sigma}, c_{j,\tau}\} = \left\{c^\dagger_{i,\sigma}, c^\dagger_{j,\tau}\right\} = 0 \tag{2.5}$$

and

$$\left\{c_{i,\sigma}, c^\dagger_{j,\tau}\right\} = \delta_{ij}\delta_{\sigma\tau} \tag{2.6}$$

for any $i, j \in \Lambda$ and $\sigma, \tau = \uparrow, \downarrow$. The number operator $n_{i,\sigma}$ is defined as $n_{i,\sigma} = c^\dagger_{i,\sigma} c_{i,\sigma}$. We denote by N_e the number of electrons in Λ and by Φ_0 a state with no electrons.

We assume that repulsive interactions between electrons on D-sites are infinitely large and that each of these sites is at most singly occupied. So we consider the Hilbert space $\mathbf{H}_{N_e}$ spanned by the linearly independent states of the form

$$\Psi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow) = \left(\prod_{x\in A} c^\dagger_{x,\sigma_x}\right)\left(\prod_{u\in B_\uparrow} c^\dagger_{u,\uparrow}\right)\left(\prod_{u\in B_\downarrow} c^\dagger_{u,\downarrow}\right)\Phi_0 \tag{2.7}$$

with arbitrary subsets $A \subset D, B_\uparrow, B_\downarrow \subset P$ such that $|A| + |B_\uparrow| + |B_\downarrow| = N_e$. Here $\boldsymbol{\sigma}_A$ is shorthand for a spin configuration $(\sigma_x)_{x\in A}$.

To define the Hamiltonian, we introduce the following new fermion operators:

$$a_{u,\sigma} = c_{u,\sigma} + \alpha \sum_{x\in D_u} c_{x,\sigma} \qquad \text{for} \quad u \in P \tag{2.8}$$

$$b_{x,\sigma} = c_{x,\sigma} + \beta \sum_{u\in P_x} c_{u,\sigma} \qquad \text{for} \quad x \in D \tag{2.9}$$

$$d_{x,\sigma} = \sum_{u\in P_x} \mathrm{e}^{-2\mathrm{i}Q\cdot u} a_{u,\sigma} \qquad \text{for} \quad x \in D \tag{2.10}$$

where α and β are real parameters with $\alpha \neq -\beta$ and $Q = (0, \pi)$. We also introduce operators $n^a_{u,\sigma} = a^\dagger_{u,\sigma} a_{u,\sigma}$, $n^b_{x,\sigma} = b^\dagger_{x,\sigma} b_{x,\sigma}$ and $n^d_{x,\sigma} = d^\dagger_{x,\sigma} d_{x,\sigma}$. Then, by using these operators, we define

$$H_0 = \mathcal{P}_D \left(t \sum_{x\in D} \sum_{\sigma=\uparrow,\downarrow} n^b_{x,\sigma} + s \sum_{u\in P} \sum_{\sigma=\uparrow,\downarrow} n_{u,\sigma} \right) \mathcal{P}_D \tag{2.11}$$

$$H_{\mathrm{int},1} = \mathcal{P}_D \left(-V_1 \sum_{x\in D} \sum_{\sigma=\uparrow,\downarrow} n^d_{x,-\sigma} n^b_{x,\sigma} - W_1 \sum_{x\in D} \sum_{l=1,2} \sum_{\sigma=\uparrow,\downarrow} (n_{x+\delta_l,\sigma} + n_{x-\delta_l,\sigma}) n^b_{x,\sigma} \right) \mathcal{P}_D \tag{2.12}$$

and

$$H_{\mathrm{int},2} = \mathcal{P}_D \left(-V_2 \sum_{u\in P} \sum_{\sigma=\uparrow,\downarrow} n^a_{u,-\sigma} n_{u,\sigma} - W_2 \sum_{u\in P} \sum_{x\in D_u} \sum_{\sigma=\uparrow,\downarrow} n_{x,\sigma} n_{u,\sigma} \right) \mathcal{P}_D \tag{2.13}$$

where t, s, V_1, W_1, V_2 and W_2 are real parameters, and $-\sigma$ denotes the spin opposite to σ. The projection operator $\mathcal{P}_D$ which eliminates states with doubly occupied D-sites is defined by

$$\mathcal{P}_D = \prod_{x\in D} \mathcal{P}_x \tag{2.14}$$

with

$$\mathcal{P}_x = (1 - n_{x,\uparrow} n_{x,\downarrow}). \tag{2.15}$$

The Hamiltonian H_0 is rewritten as

$$H_0 = \mathcal{P}_D \left(\sum_{i,j\in\Lambda} t_{ij} c^\dagger_{i,\sigma} c_{j,\sigma} \right) \mathcal{P}_D \tag{2.16}$$

with

$$t_{ij} = \begin{cases} t & \text{if } i = j \in D \\ 2\beta^2 t + s & \text{if } i = j \in P \\ \beta t & \text{if } |i - j| = 1/2 \\ \beta^2 t & \text{if } i \neq j,\ i,\ j \in P_x \text{ for some } x \in D \\ 0 & \text{otherwise} \end{cases} \tag{2.17}$$

(see figure 1) and it describes quantum mechanical motion of electrons feeling infinitely large repulsive interactions at D-sites. The Hamiltonians $H_{\mathrm{int},1}$ and $H_{\mathrm{int},2}$ correspond to correlated-hopping, pair-hopping, charge–charge and spin–spin interactions. We can observe this by rewriting, for example, $H_{\mathrm{int},2}$ as

$$\begin{aligned} H_{\mathrm{int},2} = \mathcal{P}_D \Bigg(\sum_{u\in P} \Bigg(& -\alpha^2 V_2 \sum_{x,y\in D_u; x\neq y} \sum_{\sigma=\uparrow,\downarrow} c^\dagger_{x,-\sigma} c_{y,-\sigma} n_{u,\sigma} \\ & - \alpha V_2 \sum_{x\in D_u} \sum_{\sigma=\uparrow,\downarrow} \left(c^\dagger_{x,-\sigma} c_{u,-\sigma} + c^\dagger_{u,-\sigma} c_{x,-\sigma} \right) n_{u,\sigma} - 2V_2 n_{u,\uparrow} n_{u,\downarrow} \\ & - \frac{1}{2}(\alpha^2 V_2 + W_2) \sum_{x\in D_u} n_x n_u + 2(\alpha^2 V_2 - W_2) \sum_{x\in D_u} S^{(3)}_x S^{(3)}_u \Bigg) \Bigg) \mathcal{P}_D \end{aligned} \tag{2.18}$$

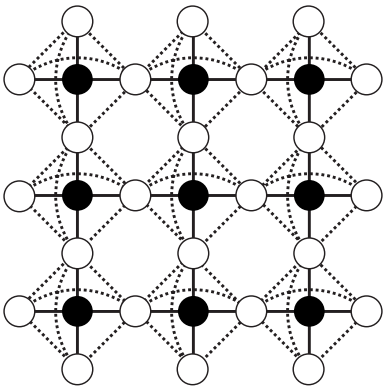

Figure 1. The lattice structure and the hopping matrix elements of H_0. The filled circles and the open circles correspond to D-sites and P-sites, respectively. The on-site potential of D-sites is t and that of P-sites is $2\beta^2 t + s$. The solid lines and the dashed lines represent the hopping matrix elements, βt and $\beta^2 t$, respectively.

where $n_i = n_{i,\uparrow} + n_{i\downarrow}$ and $S_i^{(3)} = (n_{i,\uparrow} - n_{i\downarrow})/2$. The Hamiltonian $H_{\mathrm{int},1}$ can be rewritten as well, but that has a somewhat complicated form. We note that the interactions $H_{\mathrm{int},1}$ and $H_{\mathrm{int},2}$ conserve the electron number and the eigenvalue of $\sum_{i\in\Lambda} S_i^{(3)}$, but they do not possess the spin $SU(2)$ symmetry.

In this paper we consider the Hamiltonian

$$H = H_0 + H_{\mathrm{int},1} + H_{\mathrm{int},2} \tag{2.19}$$

with the parameters satisfying $t = 4(1+\alpha^2)V_1$, $W_1 = \alpha^2 V_1$ $s = (1+2\alpha^2)V_2$, $W_2 = \alpha^2 V_2$ and $V_1, V_2 > 0$. In this case, the direct spin–spin interaction terms vanish and the charge–charge interactions are attractive.

In our construction of exact ground states, it is crucial to rewrite H as

$$H = \sum_{x\in D} H_x + \sum_{u\in P} H_u \tag{2.20}$$

with

$$H_x = \mathcal{P}_D \left(V_1 \sum_{\sigma=\uparrow,\downarrow} b_{x,\sigma}^\dagger d_{x,-\sigma} \mathcal{P}_D d_{x,-\sigma}^\dagger b_{x,\sigma} \right) \mathcal{P}_D \tag{2.21}$$

for $x \in D$ and

$$H_u = \mathcal{P}_D \left(V_2 \sum_{\sigma=\uparrow,\downarrow} c_{u,\sigma}^\dagger a_{u,-\sigma} \mathcal{P}_D a_{u,-\sigma}^\dagger c_{u,\sigma} \right) \mathcal{P}_D \tag{2.22}$$

for $u \in P$. Here we used the operator identities [2, 14]

$$\mathcal{P}_D c_{i,\sigma} \mathcal{P}_D c_{j,\sigma}^\dagger \mathcal{P}_D = \begin{cases} -\mathcal{P}_D c_{j,\sigma}^\dagger c_{i,\sigma} \mathcal{P}_D & \text{if } i \neq j \\ \mathcal{P}_D (1 - n_{i,\uparrow} - n_{i,\downarrow}) \mathcal{P}_D & \text{if } i = j \in D \\ \mathcal{P}_D (1 - n_{i,\sigma}) \mathcal{P}_D & \text{if } i = j \in P \end{cases} \tag{2.23}$$

and

$$\mathcal{P}_x c_{i,\sigma} \mathcal{P}_x = c_{i,\sigma} \mathcal{P}_x \tag{2.24}$$

$$\mathcal{P}_x c^\dagger_{i,\sigma} \mathcal{P}_x = \mathcal{P}_x c^\dagger_{i,\sigma}. \tag{2.25}$$

Operator identities (2.24) and (2.25) follow from straightforward calculations, and (2.23) is derived by using $n_{x,\uparrow} n_{x,\downarrow} \mathcal{P}_x = 0$ and

$$c_{i,\sigma}\mathcal{P}_x = \begin{cases} (1-n_{x,-\sigma})c_{x,\sigma} = \mathcal{P}_x(1-n_{x,-\sigma})c_{x,\sigma} & \text{if } i = x \\ \mathcal{P}_x c_{i,\sigma} & \text{otherwise} \end{cases} \tag{2.26}$$

$$\mathcal{P}_x c^\dagger_{i,\sigma} = \begin{cases} c^\dagger_{x,\sigma}(1-n_{x,-\sigma}) = c^\dagger_{x,\sigma}(1-n_{x,-\sigma})\mathcal{P}_x & \text{if } i = x \\ c^\dagger_{i,\sigma}\mathcal{P}_x & \text{otherwise} \end{cases} \tag{2.27}$$

which also follow from simple calculations. We note that H_i with $i \in \Lambda$ are positive semidefinite operators and thus a zero-energy state for all H_i (if it exists) is a ground state.

Let us define

$$\zeta^\dagger = \sum_{u\in P} \mathrm{e}^{2\mathrm{i}Q\cdot u} a^\dagger_{u,\uparrow} a^\dagger_{u,\downarrow} \tag{2.28}$$

which corresponds to a singlet two-electron state. The main result in this paper is the following theorem:

Theorem 2.1. *Fix the electron number N_e and suppose that $N_\mathrm{e} \leqslant 2|P|$. When N_e is even, the ground state $\Phi_{\mathrm{G},N_\mathrm{e}}$ of H is unique and is given by*

$$\Phi_{\mathrm{G},N_\mathrm{e}} = \mathcal{P}_D(\zeta^\dagger)^{\frac{N_\mathrm{e}}{2}}\Phi_0 = (\mathcal{P}_D\zeta^\dagger)^{\frac{N_\mathrm{e}}{2}}\Phi_0 \tag{2.29}$$

which satisfies $H\Phi_{\mathrm{G},N_\mathrm{e}} = 0$. For odd N_e, the ground-state energy is positive.

It is noted that the ground state energy for odd N_e may converge to zero as $L \to \infty$. Whether this is the case or not should be clarified in a future study.

3. Some remarks

It is easy to verify the relation

$$\zeta^\dagger = \sum_{x\in D}\sum_{l=1,2} \mathrm{e}^{\mathrm{i}Q\cdot\delta_l}\left(\alpha^2 f^\dagger_{x,x+\delta_l} + \alpha f^\dagger_{x,x+\delta_l/2} + \alpha f^\dagger_{x,x-\delta_l/2}\right) + \sum_{u\in P} \mathrm{e}^{2\mathrm{i}Q\cdot u} c^\dagger_{u,\uparrow} c^\dagger_{u,\downarrow} \tag{3.1}$$

where $f^\dagger_{i,j} = c^\dagger_{i,\uparrow}c^\dagger_{j,\downarrow} + c^\dagger_{j,\uparrow}c^\dagger_{i,\downarrow}$ is the creation operator for the valence bond (singlet pair) on sites i and j. Furthermore, the projection operator $\mathcal{P}_D$ eliminates the terms of the form

$$f^\dagger_{x,i} f^\dagger_{x,j} \cdots \Phi_0 = -c^\dagger_{x,\uparrow} c^\dagger_{x,\downarrow} f^\dagger_{i,j} \cdots \Phi_0 \tag{3.2}$$

when we express $\Phi_{\mathrm{G},N_\mathrm{e}}$ by using $f^\dagger_{i,j}$ and $c^\dagger_{u,\uparrow}c^\dagger_{u,\downarrow}$. Thus the ground state $\Phi_{\mathrm{G},N_\mathrm{e}}$ has the form of the RVB state which is a linear combination of the products of on-site singlet pairs $c^\dagger_{u,\uparrow}c^\dagger_{u,\downarrow}$ in P and the valence bonds $f^\dagger_{x,x+\delta_l}$ and $f^\dagger_{x,x\pm\delta_l/2}$. It is noted that the weight for the singlet pairs $f^\dagger_{x,x+\delta_l}$ on the nearest-neighbour D-sites becomes relatively large for $1/\alpha \ll 1$.

The term $-V_1 n^d_{x,-\sigma} n^b_{x,\sigma}$ in interaction Hamiltonian $H_{\mathrm{int},1}$ is rewritten as

$$-V_1 n^d_{x,-\sigma} n^b_{x,\sigma} = -4(1+\alpha^2)(1+4\beta^2)V_1 + 4(1+\alpha^2)V_1 b_{x,\sigma} b^\dagger_{x,\sigma} + V_1 d_{x,-\sigma} b^\dagger_{x,\sigma} b_{x,\sigma} d^\dagger_{x,-\sigma}. \tag{3.3}$$

From this expression, we find that $-V_1 n^d_{x,-\sigma} n^b_{x,\sigma}$ is bounded below by $-4(1+\alpha^2)(1+4\beta^2)V_1$ and furthermore that it is attained by states of the form $b^\dagger_{x,\sigma} d^\dagger_{x,-\sigma} \cdots \Phi_0$. Therefore this

interaction term is describing an attractive and magnetic interaction between localized single-electron states corresponding to fermion operators $b_{x,\sigma}$ and $d_{x,\sigma}$. The other terms can be rewritten into similar forms and interpreted as attractive and magnetic interactions as well. Our model says that attractive magnetic interactions between localized states together with the strong on-site repulsion can stabilize the condensed RVB states consisting of local singlet pairs.

The explicit expression of the ground states makes it possible to evaluate order parameters. We first note that, in the case of $\alpha = 0$, $\Phi_{\mathrm{G},N_\mathrm{e}}$ is reduced to the η-pairing state on the P-sites, and therefore the ground states have off-diagonal long-range order.

For $\alpha \neq 0$ we can prove the existence of off-diagonal long-range order as follows. Let us introduce order parameters

$$\Delta_{1,\Lambda} = \mathcal{P}_D \left(\frac{1}{|P|} \sum_{u \in P} \mathrm{e}^{-2\mathrm{i}Q\cdot u} c_{u,\downarrow} c_{u,\uparrow} \right) \mathcal{P}_D \tag{3.4}$$

and

$$\Delta_{2,\Lambda} = \mathcal{P}_D \left(\frac{1}{|P|} \sum_{u \in P} \mathrm{e}^{-2\mathrm{i}Q\cdot u} a_{u,\downarrow} a_{u,\uparrow} \right) \mathcal{P}_D = \frac{1}{|P|} \mathcal{P}_D \zeta \mathcal{P}_D. \tag{3.5}$$

Noting the anticommutation relation

$$\left\{ c_{u,\sigma}, a^\dagger_{u',\sigma} \right\} = \delta_{uu'} \tag{3.6}$$

for $u, u' \in P$, one finds the commutation relation

$$[\Delta_{1,\Lambda}, \mathcal{P}_D \zeta^\dagger \mathcal{P}_D] = \mathcal{P}_D \left(\frac{1}{|P|} \sum_{u \in P} \left(1 - a^\dagger_{u,\uparrow} c_{u,\uparrow} - a^\dagger_{u,\downarrow} c_{u,\downarrow} \right) \right) \mathcal{P}_D \tag{3.7}$$

and

$$\mathcal{P}_D \sum_{u \in P} \left(1 - a^\dagger_{u,\uparrow} c_{u,\uparrow} - a^\dagger_{u,\downarrow} c_{u,\downarrow} \right) \mathcal{P}_D \Phi_{\mathrm{G},N_\mathrm{e}} = (|P| - N_\mathrm{e}) \Phi_{\mathrm{G},N_\mathrm{e}}. \tag{3.8}$$

These two relations lead to

$$\Delta_{1,\Lambda} \Phi_{\mathrm{G},N_\mathrm{e}} = \left(\frac{N_\mathrm{e}}{2} - \frac{1}{|P|} \frac{N_\mathrm{e}}{2} \left(\frac{N_\mathrm{e}}{2} - 1 \right) \right) \Phi_{\mathrm{G},N_\mathrm{e}-2}. \tag{3.9}$$

Therefore we obtain

$$\left\langle \Delta^\dagger_{2,\Lambda} \Delta_{1,\Lambda} \right\rangle_{\Lambda,N_\mathrm{e}} = \frac{1}{|P|} \left(\frac{N_\mathrm{e}}{2} - \frac{1}{|P|} \frac{N_\mathrm{e}}{2} \left(\frac{N_\mathrm{e}}{2} - 1 \right) \right) \tag{3.10}$$

where the expectation value $\langle \cdots \rangle_{\Lambda,N_\mathrm{e}}$ is defined by

$$\langle \cdots \rangle_{\Lambda,N_\mathrm{e}} = \frac{\left(\Phi_{\mathrm{G},N_\mathrm{e}}, \ldots \Phi_{\mathrm{G},N_\mathrm{e}} \right)}{\left(\Phi_{\mathrm{G},N_\mathrm{e}}, \Phi_{\mathrm{G},N_\mathrm{e}} \right)}. \tag{3.11}$$

We write $\mu_{\Lambda,N_\mathrm{e}}$ for the right-hand side of (3.10). Then, by using the Schwarz inequality

$$\left| \left\langle \Delta^\dagger_{2,\Lambda} \Delta_{1,\Lambda} \right\rangle_{\Lambda,N_\mathrm{e}} \right|^2 \leqslant \left\langle \Delta^\dagger_{2,\Lambda} \Delta_{2,\Lambda} \right\rangle_{\Lambda,N_\mathrm{e}} \left\langle \Delta^\dagger_{1,\Lambda} \Delta_{1,\Lambda} \right\rangle_{\Lambda,N_\mathrm{e}} \tag{3.12}$$

and inequalities $\left\langle \Delta^\dagger_{1,\Lambda} \Delta_{1,\Lambda} \right\rangle_{\Lambda,N_\mathrm{e}} \leqslant 1$ and $\left\langle \Delta^\dagger_{2,\Lambda} \Delta_{2,\Lambda} \right\rangle_{\Lambda,N_\mathrm{e}} \leqslant (1+2\alpha)^2$, we find

$$\frac{\mu^2_{\Lambda,N_\mathrm{e}}}{(1+2\alpha^2)^2} \leqslant \left\langle \Delta^\dagger_{1,\Lambda} \Delta_{1,\Lambda} \right\rangle_{\Lambda,N_\mathrm{e}} \leqslant 1 \tag{3.13}$$

and

$$\mu^2_{\Lambda,N_\mathrm{e}} \leqslant \left\langle \Delta^\dagger_{2,\Lambda} \Delta_{2,\Lambda} \right\rangle_{\Lambda,N_\mathrm{e}} \leqslant (1+2\alpha^2)^2. \tag{3.14}$$

Let $\{\Phi^{(\nu)}_{\mathrm{G},L}\}_{L=0}^{\infty}$ be a sequence of ground states $\Phi_{\mathrm{G},N_\mathrm{e}}$ on the lattice with side length L such that the electron filling $N_\mathrm{e}/(2|\Lambda|) = N_\mathrm{e}/(6L^2)$ converges to ν as $L \to \infty$, and let $\{\langle\cdots\rangle^{(\nu)}_L\}_{L=0}^{\infty}$ be a sequence of expectation values with respect to $\Phi^{(\nu)}_{\mathrm{G},L}$. Then from inequalities (3.13) and (3.14) we have

$$\liminf_{L\to\infty} \left\langle \Delta^\dagger_{1,\Lambda}\Delta_{1,\Lambda}\right\rangle^{(\nu)}_L \geqslant \frac{\mu_\nu^2}{(1+2\alpha^2)^2} \tag{3.15}$$

and

$$\liminf_{L\to\infty} \left\langle \Delta^\dagger_{2,\Lambda}\Delta_{2,\Lambda}\right\rangle^{(\nu)}_L \geqslant \mu_\nu^2 \tag{3.16}$$

where $\mu_\nu = \frac{3\nu}{2}\left(1-\frac{3\nu}{2}\right)$, which imply the existence of off-diagonal long-range order for $0 < \nu < 2/3$.

Now we can construct a ground state with explicit electron-number symmetry breaking [15]. Let us define

$$\Phi'_{\mathrm{G},N_\mathrm{e}} = \Phi_{\mathrm{G},N_\mathrm{e}} + \frac{\Delta^\dagger_{2,\Lambda}}{\sqrt{\left\langle \Delta_{2,\Lambda}\Delta^\dagger_{2,\Lambda}\right\rangle_{\Lambda,N_\mathrm{e}}}}\Phi_{\mathrm{G},N_\mathrm{e}} \tag{3.17}$$

which is also a zero-energy state. We note that the limit infimum of $\left\langle \Delta_{2,\Lambda}\Delta^\dagger_{2,\Lambda}\right\rangle^{(\nu)}_L$ is bounded below by μ_ν^2, since

$$\left\langle \Delta_{1,\Lambda}\Delta^\dagger_{2,\Lambda}\right\rangle_{\Lambda,N_\mathrm{e}} = \left\langle \Delta^\dagger_{2,\Lambda}\Delta_{1,\Lambda}\right\rangle_{\Lambda,N_\mathrm{e}} + \frac{1}{|P|^2}(|P| - N_\mathrm{e}) \tag{3.18}$$

which follows from (3.7). Since two states with different electron numbers are orthogonal, we find that

$$\langle\Delta_{2,\Lambda}\rangle'_{\Lambda,N_\mathrm{e}} = \frac{\left(\Phi'_{\mathrm{G},N_\mathrm{e}},\, \Delta_{2,\Lambda}\Phi'_{\mathrm{G},N_\mathrm{e}}\right)}{\left(\Phi'_{\mathrm{G},N_\mathrm{e}},\, \Phi'_{\mathrm{G},N_\mathrm{e}}\right)} = \frac{1}{2}\sqrt{\left\langle \Delta_{2,\Lambda}\Delta^\dagger_{2,\Lambda}\right\rangle_{\Lambda,N_\mathrm{e}}}. \tag{3.19}$$

Therefore, the limit infimum of the sequence of $\langle\Delta_{2,\Lambda}\rangle'_{\Lambda,N_\mathrm{e}}$ obtained by using $\{\Phi^{(\nu)}_{\mathrm{G},L}\}_{L=0}^{\infty}$ with (3.17) is bounded below by $\mu_\nu/2$, which implies electron-number symmetry breaking.

We finally remark on extensions of the present model. It is possible to construct similar models in one, three and more dimensions by the same method. The model in three or more dimensions may exhibit a finite temperature phase transition. It is also possible to construct models whose ground states are written as $(\zeta^\dagger)^{N_\mathrm{e}/2}\Phi_0$. The details will appear elsewhere.

4. Proof

Proof of theorem 2.1. We first prove that $\Phi_{\mathrm{G},N_\mathrm{e}}$ in (2.29) is a zero-energy state for all H_i and thus is a ground state. Using the anticommutation relations (3.6) and

$$\left\{c_{x,\sigma}, a^\dagger_{u,\sigma}\right\} = \alpha\chi[u \in P_x] \tag{4.1}$$

for $x \in D$, where χ['event'] takes 1 if 'event' is true and takes 0 otherwise, one finds the following two commutation relations:

$$[c_{u,\sigma}, \zeta^\dagger] = \sigma\, \mathrm{e}^{2\mathrm{i}Q\cdot u} a^\dagger_{u,-\sigma} \tag{4.2}$$

for $u \in P$ and

$$[c_{x,\sigma}, \zeta^\dagger] = \sigma\alpha \sum_{u\in P_x} \mathrm{e}^{2\mathrm{i}Q\cdot u} a^\dagger_{u,-\sigma} \tag{4.3}$$

for $x \in D$. (On the right-hand sides of (4.2) and (4.3), the coefficients $\sigma = \uparrow$ and $\downarrow$ are regarded as +1 and −1, respectively. We will use this convention in the following.) By using operator identities (2.25), (2.26) and commutation relation (4.2), we find

$$\begin{aligned}\left(\mathcal{P}_D a^\dagger_{u,-\sigma} c_{u,\sigma}\right)\mathcal{P}_D\zeta^\dagger &= \mathcal{P}_D a^\dagger_{u,-\sigma} c_{u,\sigma}\zeta^\dagger \\ &= \mathcal{P}_D a^\dagger_{u,-\sigma}\left(\zeta^\dagger c_{u,\sigma} + \sigma\, \mathrm{e}^{2\mathrm{i}Q\cdot u} a^\dagger_{u,-\sigma}\right) \\ &= \mathcal{P}_D\zeta^\dagger\left(\mathcal{P}_D a^\dagger_{u,-\sigma} c_{u,\sigma}\right). \end{aligned} \tag{4.4}$$

To get the third line, we used $\left(a^\dagger_{u,-\sigma}\right)^2 = 0$. This implies that $\left(\mathcal{P}_D a^\dagger_{u,-\sigma} c_{u,\sigma}\right)(\mathcal{P}_D\zeta^\dagger)^{N_\mathrm{e}/2}\Phi_0 = 0$ and thus $H_u\Phi_{\mathrm{G},N_\mathrm{e}} = 0$ for all $u \in P$. Similarly, by using (2.25), (2.26) and commutation relations (4.2), (4.3) we have

$$\begin{aligned}\left(\mathcal{P}_D d^\dagger_{x,-\sigma}\sum_{u\in P_x} c_{u,\sigma}\right)\mathcal{P}_D\zeta^\dagger &= \left(\mathcal{P}_D d^\dagger_{x,-\sigma}\sum_{u\in P_x} c_{u,\sigma}\right)\zeta^\dagger \\ &= \mathcal{P}_D d^\dagger_{x,-\sigma}\zeta^\dagger\sum_{u\in P_x} c_{u,\sigma} + \sigma\mathcal{P}_D d^\dagger_{x,-\sigma}\sum_{u\in P_x}\mathrm{e}^{2\mathrm{i}Q\cdot u}a^\dagger_{u,-\sigma} \\ &= \mathcal{P}_D\zeta^\dagger\left(\mathcal{P}_D d^\dagger_{x,-\sigma}\sum_{u\in P_x} c_{u,\sigma}\right) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\left(\mathcal{P}_D d^\dagger_{x,-\sigma} c_{x,\sigma}\right)\mathcal{P}_D\zeta^\dagger &= \mathcal{P}_D(1-n_{x,-\sigma})d^\dagger_{x,-\sigma}c_{x,\sigma}\zeta^\dagger \\ &= \mathcal{P}_D(1-n_{x,-\sigma})d^\dagger_{x,-\sigma}\zeta^\dagger c_{x,\sigma} + \sigma\alpha\mathcal{P}_D(1-n_{x,-\sigma})d^\dagger_{x,-\sigma}\sum_{u\in P_x}\mathrm{e}^{2\mathrm{i}Q\cdot u}a^\dagger_{u,-\sigma} \\ &= \mathcal{P}_D(1-n_{x,-\sigma})\zeta^\dagger\left(\mathcal{P}_D d^\dagger_{x,-\sigma}c_{x,\sigma}\right). \end{aligned} \tag{4.6}$$

Here we also used $\left\{c_{x,-\sigma}, d^\dagger_{x,-\sigma}\right\} = 0$ and $\left(d^\dagger_{x,-\sigma}\right)^2 = 0$. It follows from (4.5) and (4.6) that $\left(\mathcal{P}_D d^\dagger_{x,-\sigma} b_{x,\sigma}\right)(\mathcal{P}_D\zeta^\dagger)^{N_\mathrm{e}/2}\Phi_0 = 0$, i.e., $H_x\Phi_{\mathrm{G},N_\mathrm{e}} = 0$ for all $x \in D$. Therefore we conclude that $H\Phi_{\mathrm{G},N_\mathrm{e}} = 0$.

The proof for the other statements in the theorem relies on the following lemma, which will be proved later.

Lemma 4.1. *Any zero-energy state Φ for H in the Hilbert space $\bigoplus_{N_\mathrm{e}=0}^{2|P|}\mathbf{H}_{N_\mathrm{e}}$ is written as*

$$\Phi = \sum_{B\subset P}\phi(B)\mathcal{P}_D\left(\prod_{u\in B}a^\dagger_{u,\uparrow}\right)\left(\prod_{u\in B}a^\dagger_{u,\downarrow}\right)\Phi_0 \tag{4.7}$$

where $\phi(B)$ are real coefficients. Furthermore, coefficients satisfy $|\phi(B)| = |\phi(B')|$ for any $B, B' \subset P$ such that $|B| = |B'|$.

It immediately follows from lemma 4.1 that the ground-state energy for odd N_e is always positive. The remaining task is to prove the uniqueness. Suppose that there are two zero-energy states for fixed even N_e. Then, an arbitrary linear combination of these states is also a zero-energy state, which should satisfy lemma 4.1. However, we can make a suitable linear combination so that a coefficient $\phi(B_0)$ for a subset B_0 will be vanishing, and this leads to the conclusion that all the other coefficients are also vanishing. This contradicts the assumption, and therefore the ground state is unique. This completes the proof of theorem 2.1. □

Before proceeding to the proof of lemma 4.1, we prove the following lemma.

Lemma 4.2. *For $A \subset D$, spin configuration $\boldsymbol{\sigma}_A$ and $B_\uparrow, B_\downarrow \subset P$, define*

$$\Phi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow) = \left(\prod_{x\in A} c^\dagger_{x,\sigma_x}\right)\left(\prod_{u\in B_\uparrow} a^\dagger_{u,\uparrow}\right)\left(\prod_{u\in B_\downarrow} a^\dagger_{u,\downarrow}\right)\Phi_0. \tag{4.8}$$

Then, the states $\mathcal{P}_D\Phi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow)$ are linearly independent, and the collection of these states with $|A| + |B_\uparrow| + |B_\downarrow| = N_e$ spans the Hilbert space $\mathbf{H}_{N_e}$.

Proof of lemma 4.2. Consider a linear combination

$$\Phi = \sum_{A\subset D}\sum_{\boldsymbol{\sigma}_A}\sum_{B_\uparrow, B_\downarrow\subset P} \phi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow)\mathcal{P}_D\Phi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow) \tag{4.9}$$

where $\sum_{\boldsymbol{\sigma}_A}$ means the sum taken over all spin configurations. We suppose that $\Phi = 0$. From the anticommutation relation (3.6) it follows that the inner product

$$(\Psi(A', \boldsymbol{\sigma}'_{A'}; B'_\uparrow, B'_\downarrow), \mathcal{P}_D\Phi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow)) \tag{4.10}$$

is zero if both $B'_\uparrow \subset B_\uparrow$ and $B'_\downarrow \subset B_\downarrow$ do not hold. (See (2.7) for the definition of $\Psi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow)$.) We furthermore have that

$$(\Psi(A', \boldsymbol{\sigma}'_{A'}; B_\uparrow, B_\downarrow), \mathcal{P}_D\Phi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow)) = 0 \tag{4.11}$$

for $A' \neq A$ and that

$$(\Psi(A, \boldsymbol{\sigma}'_A; B_\uparrow, B_\downarrow), \mathcal{P}_D\Phi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow)) = \chi[\sigma'_x = \sigma_x \text{ for all } x \in A]. \tag{4.12}$$

Thus we obtain from $(\Psi(A, \boldsymbol{\sigma}_A; P, P), \Phi) = 0$ that $\phi(A, \boldsymbol{\sigma}_A; P, P) = 0$ for any $A \subset D$ and spin configuration $\boldsymbol{\sigma}_A$. Then, examining

$$(\Psi(A, \boldsymbol{\sigma}_A; P\backslash\{u\}, P), \Phi) = 0 \tag{4.13}$$

for $u \in P$, we find that $\phi(A, \boldsymbol{\sigma}_A; P\backslash\{u\}, P) = 0$. Similarly, examining

$$(\Psi(A, \boldsymbol{\sigma}_A; P\backslash\bar{B}_\uparrow, P\backslash\bar{B}_\downarrow), \Phi) = 0 \tag{4.14}$$

for $\bar{B_\uparrow}, \bar{B}_\downarrow \subset P$ with $|\bar{B}_\uparrow|, |\bar{B}_\downarrow| = 0, 1, 2, \ldots |P|$ repeatedly, we conclude that all the coefficients $\phi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow)$ are vanishing. This proves the first claim, and the second claim is now trivial. □

Proof of lemma 4.1. Suppose that Φ is an arbitrary zero-energy state for H in $\bigoplus_{N_e=0}^{2|P|}\mathbf{H}_{N_e}$. By using the basis states $\mathcal{P}_D\Phi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow)$ with $|A| + |B_\uparrow| + |B_\downarrow| \leqslant 2|P|$, we represent Φ as

$$\Phi = \sum_{A\subset D}\sum_{\boldsymbol{\sigma}_A}\sum_{B_\uparrow, B_\downarrow\subset P} \phi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow)\mathcal{P}_D\Phi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow). \tag{4.15}$$

The sum is restricted to $|A| + |B_\uparrow| + |B_\downarrow| \leqslant 2|P|$, but, here and in the following, we do not write down this restriction for notational simplicity.

Since the local Hamiltonian H_i is the sum of two positive semidefinite operators, zero-energy state Φ for H must be annihilated by these operators for any $i \in \Lambda$.

We first examine the case $i = u \in P$. In this case the condition $H_u\Phi = 0$ is equivalent to

$$\mathcal{P}_D a^\dagger_{u,-\sigma} c_{u,\sigma}\mathcal{P}_D\Phi = 0 \tag{4.16}$$

for $\sigma = \uparrow, \downarrow$. For $\sigma = \uparrow$ the left-hand side of (4.16) becomes

$$\sum_{A\subset D}\sum_{\boldsymbol{\sigma}_A}\sum_{B_\uparrow, B_\downarrow\subset P} \mathbf{sgn}[u; B_\uparrow, B_\downarrow]\chi[u \in B_\uparrow]\chi[u \notin B_\downarrow]$$
$$\times \phi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow)\mathcal{P}_D\Phi(A, \boldsymbol{\sigma}_A; B_\uparrow\backslash\{u\}, B_\downarrow \cup \{u\}) \tag{4.17}$$

where $\mathbf{sgn}[u; B_\uparrow, B_\downarrow]$ is a sign factor arising from exchanges of fermion operators. Since all the terms in (4.17) are linearly independent, we have the condition $\phi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow) = 0$ for $B_\uparrow, B_\downarrow$ such that $u \in B_\uparrow$ and $u \notin B_\downarrow$. A similar calculation for $\sigma = \downarrow$ yields $\phi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow) = 0$ for $B_\uparrow, B_\downarrow$ such that $u \notin B_\uparrow$ and $u \in B_\downarrow$. Since these conditions must be satisfied for all $u \in P$, we obtain $\phi(A, \boldsymbol{\sigma}_A; B_\uparrow, B_\downarrow) = 0$ for $B_\uparrow \neq B_\downarrow$.

So far we have shown that a zero-energy state Φ can be expanded as

$$\Phi = \sum_{A \subset D} \sum_{\boldsymbol{\sigma}_A} \sum_{B \subset P} \phi(A, \boldsymbol{\sigma}_A; B) \mathcal{P}_D \Phi(A, \boldsymbol{\sigma}_A; B, B) \tag{4.18}$$

where $\phi(A, \boldsymbol{\sigma}_A; B) = \phi(A, \boldsymbol{\sigma}_A; B, B)$. We further derive conditions on $\phi(A, \boldsymbol{\sigma}_A; B)$ from $H_i \Phi = 0$ with $i = x \in D$, which is equivalent to

$$\mathcal{P}_D d^\dagger_{x,-\sigma} b_{x,\sigma} \mathcal{P}_D \Phi = 0 \tag{4.19}$$

for $\sigma = \uparrow, \downarrow$. The left-hand side of (4.19) can be expanded by using (4.8), so that we decompose this as

$$\sum_{A' \subset D; x \notin A'} \sum_{\boldsymbol{\sigma}'_{A'}} \sum_{B'_\uparrow, B'_\downarrow \subset P} \cdots + \sum_{A' \subset D; x \in A'} \sum_{\boldsymbol{\sigma}'_{A'}} \sum_{B'_\uparrow, B'_\downarrow \subset P} \cdots \tag{4.20}$$

and write Φ_x for the first sum. Since the states in the first sum and those in the second sum are linearly independent, $\Phi_x = 0$ must be satisfied.

To obtain Φ_x for $\sigma = \uparrow$, we operate $\mathcal{P}_D d^\dagger_{x,\downarrow} b_{x,\uparrow} \mathcal{P}_D$ on the basis state $\mathcal{P}_D \Phi(A, \boldsymbol{\sigma}_A; B, B)$. Then, we have

$$(1 - n_{x,\downarrow}) \mathcal{P}_D d^\dagger_{x,\downarrow} c_{x,\uparrow} \Phi(A, \boldsymbol{\sigma}_A; B, B) + \beta \mathcal{P}_D d^\dagger_{x,\downarrow} \left(\sum_{u \in P_x} c_{u,\uparrow} \right) \Phi(A, \boldsymbol{\sigma}_A; B, B). \tag{4.21}$$

The second term in (4.21) becomes

$$\beta \sum_{u,u' \in P_x} \mathbf{sgn}[u, u'; B] \chi[u \in B] \chi[u' \notin B] \, \mathrm{e}^{2\mathrm{i}Q \cdot u'} \mathcal{P}_D \Phi(A, \boldsymbol{\sigma}_A; B \backslash \{u\}, B \cup \{u'\}) \tag{4.22}$$

where $\mathbf{sgn}[u, u'; B]$ is a sign factor arising from exchanges of fermion operators. Thus, the second term contributes to Φ_x only when $x \notin A$. The first term in (4.21) becomes

$$\begin{aligned}(\chi[x \in A] \chi[\sigma_x = \uparrow] + \chi[x \notin A]) \mathcal{P}_D d^\dagger_{x,\downarrow} c_{x,\uparrow} \Phi(A, \boldsymbol{\sigma}_A; B, B) - (\chi[x \in A] \chi[\sigma_x = \uparrow] \\ + \chi[x \notin A]) \mathcal{P}_D d^\dagger_{x,\downarrow} c^\dagger_{x,\downarrow} c_{x,\downarrow} c_{x,\uparrow} \Phi(A, \boldsymbol{\sigma}_A; B, B).\end{aligned} \tag{4.23}$$

Since site x is always occupied by an electron in the second term in (4.23), this term never contributes to Φ_x. The first term in (4.23) furthermore becomes

$$\begin{aligned}&\chi[x \in A] \chi[\sigma_x = \uparrow] \mathbf{sgn}[x; A] \sum_{u' \in P_x} \mathbf{sgn}[u'; B] \chi[u' \notin B] \\ &\quad \times \mathrm{e}^{2\mathrm{i}Q \cdot u'} \mathcal{P}_D \Phi(A \backslash \{x\}, \boldsymbol{\sigma}_{A \backslash \{x\}}; B, B \cup \{u'\}) \\ &\quad + \chi[x \in A] \chi[\sigma_x = \uparrow] \mathcal{P}_D \left(\prod_{x \in A} c^\dagger_{x,\sigma_x} \right) d^\dagger_{x,\downarrow} c_{x,\uparrow} \left(\prod_{u \in B_\uparrow} a^\dagger_{u,\uparrow} \right) \left(\prod_{u \in B_\downarrow} a^\dagger_{u,\downarrow} \right) \\ &\quad + \chi[x \notin A] \alpha \sum_{u,u' \in P_x} \mathbf{sgn}[u, u'; B] \chi[u \in B] \chi[u' \notin B] \\ &\quad \times \mathrm{e}^{2\mathrm{i}Q \cdot u'} \mathcal{P}_D \Phi(A, \boldsymbol{\sigma}_A; B \backslash \{u\}, B \cup \{u'\})\end{aligned} \tag{4.24}$$

where $\mathbf{sgn}[x;A]$ and $\mathbf{sgn}[u;B]$ are again sign factors. Since site x is always occupied by an electron in the second term in the above expression, this term does not contribute to Φ_x. Therefore we finally obtain

$$\begin{aligned}\Phi_x = &\sum_{A\subset D;x\in A}\sum_{\boldsymbol{\sigma}_A;\sigma_x=\uparrow}\sum_{B\subset P}\mathbf{sgn}[x;A]\sum_{u'\in P_x}\mathbf{sgn}[u';B]\chi[u'\notin B]\\ &\times \mathrm{e}^{2\mathrm{i}Q\cdot u'}\phi(A,\boldsymbol{\sigma}_A;B)\mathcal{P}_D\Phi(A\backslash\{x\},\boldsymbol{\sigma}_{A\backslash\{x\}};B,B\cup\{u'\})\\ &+(\alpha+\beta)\sum_{A\subset D;x\notin A}\sum_{\boldsymbol{\sigma}_A}\sum_{B\subset P}\sum_{u,u'\in P_x}\mathbf{sgn}[u,u';B]\chi[u\in B]\chi[u'\notin B]\\ &\times \mathrm{e}^{2\mathrm{i}Q\cdot u'}\phi(A,\boldsymbol{\sigma}_A;B)\mathcal{P}_D\Phi(A,\boldsymbol{\sigma}_A;B\backslash\{u\},B\cup\{u'\}).\end{aligned}\tag{4.25}$$

It is noted that the terms in the first sum are linearly independent of those in the second sum.

Choose a configuration $(A,\boldsymbol{\sigma}_A;B)$ satisfying that A contains x,σ_x in $\boldsymbol{\sigma}_A$ is $\uparrow$ and there exists $u'\in P_x$ such that $u'\notin B$. Then, by checking the coefficient of the basis state $\mathcal{P}_D\Phi(A\backslash\{x\},\boldsymbol{\sigma}_{A\backslash\{x\}};B,B\cup\{u'\})$ in (4.25), we obtain $\phi(A,\boldsymbol{\sigma}_A;B)=0$ for such a configuration. Since this and a similar result for $\sigma=\downarrow$ must hold for any $x\in D$, we obtain that

$$\phi(A,\boldsymbol{\sigma}_A;B)=0\tag{4.26}$$

if $(\cup_{x\in A}P_x)$ is not a subset of B.

Next, choose a configuration satisfying that A does not contain x and there exist $u,u'\in P_x$ such that $u\in B$ and $u'\notin B$. For this kind of configuration, by checking the coefficient of the basis state $\mathcal{P}_D\Phi(A,\boldsymbol{\sigma}_A;B\backslash\{u\},B\cup\{u'\})$ in (4.25), we have that

$$\begin{aligned}&(\alpha+\beta)(\mathbf{sgn}[u,u';B]\,\mathrm{e}^{2\mathrm{i}Q\cdot u'}\phi(A,\boldsymbol{\sigma}_A;B)\\ &\qquad+\mathbf{sgn}[u',u;B_{u\to u'}]\,\mathrm{e}^{2\mathrm{i}Q\cdot u}\phi(A,\boldsymbol{\sigma}_A;B_{u\to u'}))=0\end{aligned}\tag{4.27}$$

where $B_{u\to u'}=(B\backslash\{u\})\cup\{u'\}$. Since $\mathbf{sgn}[\cdot]$, $\mathrm{e}^{2\mathrm{i}Q\cdot u}$ and $\mathrm{e}^{2\mathrm{i}Q\cdot u'}$ give only sign factors and $\alpha+\beta$ is non-zero by definition, we obtain the (necessary) condition that, for any A not containing x and any $\boldsymbol{\sigma}_A$,

$$|\phi(A,\boldsymbol{\sigma}_A;B)|=|\phi(A,\boldsymbol{\sigma}_A;B_{u\to u'})|\tag{4.28}$$

if there exist $u,u'\in P_x$ such that $u\in B$ and $u'\notin B$.

By using conditions (4.26) and (4.28), we shall prove that $\phi(A,\boldsymbol{\sigma}_A;B)=0$ for any $A\neq\emptyset$. Suppose that $\phi(A,\boldsymbol{\sigma}_A;B)\neq 0$ for some non-empty set $A\subset D$, spin configuration $\boldsymbol{\sigma}_A$ and $B\subset P$. Here we say that u and u' in B are connected when $D_u\cap D_{u'}\neq\emptyset$ and decompose B into connected components as $B=B^1\cup\cdots\cup B^m$. We find from condition (4.26) that A is contained in $\cup_{u\in B}D_u$, and without loss of generality, we can assume that $\cup_{u\in B^1}D_u$ contains at least one site in A. For connected subset B^1 we can find at least one pair of sites $u_0,u_1\in P$ and a site $x_1\in\cup_{u\in B^1}D_u$ such that $u_0\notin B^1$, $u_1\in B^1$ and $u_0,u_1\in P_{x_1}$. When $m>1$ this is trivial, and when $m=1$ this follows from $N_\mathrm{e}=|A|+2|B|\leqslant 2|P|$. It is noted that x_1 is not in A due to condition (4.26). Then, since we can always find subsets $\{u_2,\ldots,u_n\}\subset B^1$ and $\{x_2,\ldots,x_n\}\subset\cup_{u\in B^1}D_u$ satisfying that $u_l\notin B^1_{u_l\to u_0}$, $u_{l+1}\in B^1_{u_l\to u_0}$, $u_l,u_{l+1}\in P_{x_{l+1}}$ and $x_l\notin A$ for $1\leqslant l<n$ and $x_n\in A$, the repeated use of (4.28) gives

$$|\phi(A,\boldsymbol{\sigma}_A;B)|=|\phi(A,\boldsymbol{\sigma}_A;B_{u_{n-1}\to u_0})|.\tag{4.29}$$

But, since $u_{n-1}\in P_{x_n}$ is not in $B_{u_{n-1}\to u_0}$, the coefficient $\phi(A,\boldsymbol{\sigma}_A;B_{u_{n-1}\to u_0})$ must be vanishing because of (4.26), which leads to a contradiction. Therefore the claim is proved.

From the above results we conclude that Φ is expanded as

$$\Phi=\sum_{B\subset P}\phi(B)\mathcal{P}_D\Phi(\emptyset,\boldsymbol{\sigma}_\emptyset;B,B)\tag{4.30}$$

where $\phi(B) = \phi(\emptyset, \boldsymbol{\sigma}_\emptyset; B)$. Then, using (4.28) again, we find that $|\phi(B)| = |\phi(B')|$ whenever $|B| = |B'|$, which completes the proof of lemma 4.1. □

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References

[1] Anderson P W 1987 *Science* **235** 1196
[2] Brandt U and Giesekus A 1992 *Phys. Rev. Lett.* **68** 2648
[3] Tasaki H 1993 *Phys. Rev. Lett.* **70** 3303
[4] Bares P-A and Lee P A 1994 *Phys. Rev.* B **49** 8882
[5] Yamanaka M, Honjo S, Hatsugai Y and Kohmoto M 1996 *J. Stat. Phys.* **84** 1133
[6] Yang C N 1989 *Phys. Rev. Lett.* **63** 2144
[7] Essler F H L, Korepin V E and Schoutens K 1993 *Phys. Rev. Lett.* **70** 73
[8] de Boer J, Korepin V E and Schadschneider A 1995 *Phys. Rev. Lett.* **74** 789
[9] Karnaukhov I N and Ovchinnikov A A 2002 *Phys. Rev.* B **66** 104518
[10] Ge X-Y and Gould M D 2003 *J. Phys. A: Math. Gen.* **36** 1801
[11] Shen S-Q and Qiu Z-M 1993 *Phys. Rev. Lett.* **71** 4238
[12] Shen S-Q 1996 *Phys. Rev.* B **54** 9039
[13] Lanzara A *et al* 2001 *Nature* **412** 510
[14] Tasaki H 1994 *Phys. Rev.* B **49** 7763
[15] Koma T and Tasaki H 1994 *J. Stat. Phys.* **76** 745